

# $O(N)$ –Universality Classes and the Mermin-Wagner Theorem

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We study how universality classes of  $O(N)$ –symmetric models depend continuously on the dimension  $d$  and the number of field components  $N$ . We observe, from a renormalization group perspective, how the implications of the Mermin-Wagner-Hohenberg theorem set in as we gradually deform theory space towards  $d = 2$ . For fractal dimension in the range  $2 < d < 3$  we observe, for any  $N \geq 1$ , a finite family of multi-critical effective potentials of increasing order. Apart for the  $N = 1$  case, these disappear in  $d = 2$  consistently with the Mermin-Wagner-Hohenberg theorem. Finally, we study  $O(N = 0)$ –universality classes and find an infinite family of these in two dimensions.

*Introduction.* Our modern understanding of quantum or statistical field theory is based on the ideas put forward by K. Wilson and formalized within the framework of the renormalization group (RG) [1]. This approach considers all possible theories describing the quantum or statistical fluctuations of a given set of degrees of freedom, the fields, subject only to the constraints imposed by symmetry and dimensionality. This defines what we call “theory space”, the space of all possible theories. The process of quantization on one side, or averaging on the other, is then seen as a trajectory connecting the bare action or the Hamiltonian to the full quantum or statistical effective action. This trajectory can be of finite or infinite length (with respect to the RG time); in the first case one is performing an effective field theory calculation, while in the second case one needs an ending point in theory space: this usually is a fixed-point. RG fixed-points describe scale invariant theories, i.e. theories that don’t have a characteristic length, where fluctuations on all length scales are equally important. These theories are like lighthouses that permit us to shed light on the structure of theory space. They attract or repel surrounding theories giving rise to universality, a phenomenon that underlies both non-perturbative renormalization and the understanding of continuous phase transitions. Once we find all fixed-points we can reconstruct the general (topological) properties of the RG flow in theory space and acquire a deep understanding of that class of models. A paradigmatic example of this is the  $c$ -theorem [2], which describes the RG flow between two dimensional theories.

Even if important information about two dimensional theories comes from exact results for particular lattice models, our ability to predict the universal features of two dimensional continuous phase transitions comes from the understanding of the RG structure of theory space. On the other hand, three dimensional systems are much more difficult to treat exactly; also here many analytical insights come from the RG study, while otherwise one has to resort to numerical methods. Deep insights, such as the role played by conformal symmetry in constraining statistical fluctuations, are also naturally embedded in the larger framework of RG analysis [3].

In this Letter we show how another fundamental and broad result like the Mermin-Wagner-Hohenberg theorem [4] (see [5] for a review), which states that there cannot be continuous phase transitions in  $d = 2$  systems characterized by continuous symmetries, fits in the RG

picture. We will do this by studying linear scalar  $O(N)$ –models, a class of theories that has many applications to low dimensional systems. Despite its relevance, there is no complete description of how universality classes of  $O(N)$ –models depend continuously on both  $d$  and  $N$ . In this Letter we give such a description by studying scaling solutions of the effective average action [6]. As a result we find many new  $N \geq 2$  universality classes describing multi-critical models in fractal dimension between two and three. In the  $N = 0$  case we observe an infinite number of fixed-points in  $d = 2$ , analogue to the  $N = 1$  minimal models series.

*Flow equations.* The effective average action (EAA)  $\Gamma_k[\varphi]$  is a functional that depends on the infrared scale  $k$  and that interpolates smoothly between the bare action for  $k \rightarrow \infty$  and the standard effective action for  $k \rightarrow 0$  [6]. The EAA satisfies an exact RG equation [7] that describes its dependence upon changes of scale; this equation can be used to set up a framework where to concretely implement the RG ideas discussed above. It is generally quite difficult to follow exactly the flow of the EAA through theory space and to find the relative fixed-point functionals: approximations are needed. One that retains all the important information about the structure of theory space is the one where all one-particle-irreducible (1PI) vertices of the EAA are evaluated at zero momenta. This defines the running effective potential  $U_k(\rho)$  which is a function of the  $O(N)$ –invariant  $\rho = \frac{1}{2}\varphi^2$ . The running effective potential can equivalently be defined by evaluating the EAA at a constant field configuration  $\varphi(x) \equiv \varphi$  where  $\Gamma_k[\varphi] = \Omega U_k(\rho)$ . In this approximation the RG theory space is projected onto the functional space of effective potentials. This space is still infinite dimensional and, at least at the qualitative level, all  $O(N)$ –universality classes of the full theory can be found by determining the relative scaling solutions (fixed-points).

In terms of the dimensionless (running) effective potential  $\tilde{U}_k(\tilde{\rho}) = k^{-d}U_k(\rho)$ , with  $\tilde{\rho} = k^{-(d-2+\eta)}\rho$ , a scaling solution  $\partial_t \tilde{U}_*(\tilde{\rho}) = 0$  satisfies the following ordinary differential equation [7]:

$$-(d-2+\eta)\tilde{\rho}\tilde{U}'_* + d\tilde{U}_* = c_d(N-1)\frac{1-\frac{\eta}{d+2}}{1+\tilde{U}'_*} + c_d\frac{1-\frac{\eta}{d+2}}{1+\tilde{U}_*+2\tilde{\rho}\tilde{U}''_*}, \quad (1)$$

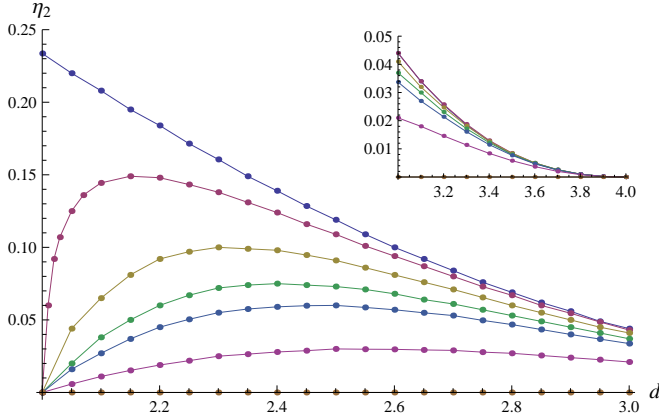


Figure 1: Anomalous dimensions  $\eta_2$  as a function of  $d$  for (from above)  $N = 1, 2, 3, 4, 5, 10, 100$ . In the inset we show the anomalous dimensions in the range  $3 < d \leq 4$ .

where  $c_d^{-1} = (4\pi)^{d/2}\Gamma(d/2 + 1)$ . The anomalous dimension  $\eta$  fixes the scaling properties of the field at a particular fixed-point; to lowest order the value of the anomalous dimension is related to the dimensionless effective potential by the following relation [6]:

$$\eta = c_d \frac{4\tilde{\rho}_0 \tilde{U}_*''(\tilde{\rho}_0)^2}{[1 + 2\tilde{\rho}_0 \tilde{U}_*''(\tilde{\rho}_0)]^2}, \quad (2)$$

where  $\tilde{\rho}_0$  is the (absolute) minimum of the dimensionless effective potential, i.e.  $U_*'(\tilde{\rho}_0) = 0$ .

Every scaling solution, together with its domain of attraction, represents a different universality class; thus by finding the solutions of the system composed of (1) and (2) one can determine all  $O(N)$ -universality classes. Differently from other implementations of RG ideas, all of the analysis can be made leaving  $d$  and  $N$  as free parameters, permitting us to study how theory space depends on these.

*Mermin-Wagner-Hohenberg theorem.* We solve the fixed-point equations (1) and (2) by the iterative method proposed in [3]. Given  $d, N$  and an initial ansatz for the anomalous dimension one can numerically integrate equation (1), subject to the initial conditions  $\tilde{U}_*(0) = \sigma$  and  $\tilde{U}_*(0) = \frac{N(d+2-\eta)c_d}{d(d+2)(1+\sigma)}$ , to find that for most values of the parameter  $\sigma$  the solution will end up in a singularity at a finite value of the dimensionless field; let's call this value  $\tilde{\rho}_s^{d,N,\eta}(\sigma)$ . Requiring a scaling solution to be well defined for any  $\tilde{\rho} \in \mathbb{R}$  restricts the admissible initial values of  $\sigma$  to a discrete set  $\{\sigma_{*,i}^{d,N,\eta}\}$ . As we will see, all these solutions correspond to multi-critical potentials of increasing order; we will use the subscript  $i$  to specify the number of minima of a scaling potential. One can construct a numerical plot of the function  $\tilde{\rho}_s^{d,N,\eta}(\sigma)$  to find the  $\sigma_{*,i}^{d,N,\eta}$  as those values where it diverges or has a “spike”. Then one inserts the solution of the ODE (1) obtained from a value  $\sigma_{*,i}^{d,N,\eta}$ , and thus corresponding to a  $i$ -critical potential, in (2) to find the next value for the anomalous dimension, and iterates the procedure until convergence. In this way

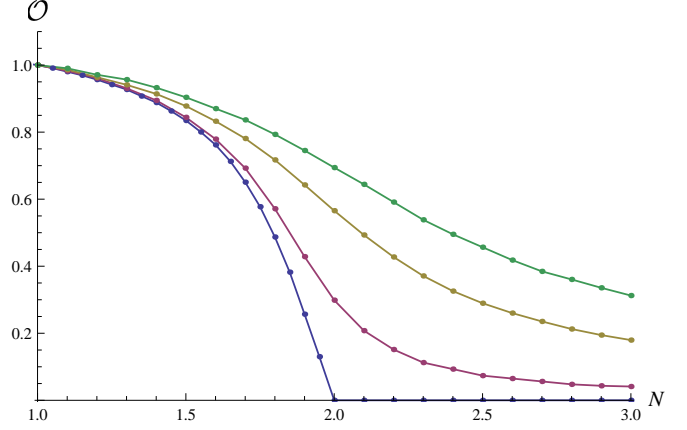


Figure 2:  $\mathcal{O}(d, N) = \eta_2(d, N)/\eta_2(d, 1)$  as a function of  $N$  for (from above)  $d = 2.1, 2.05, 2.01, 2$ . The anomalous dimension can be interpreted as the order parameter of a Mermin-Wagner-Hohenberg continuous “phase transition”. The control parameter of the transition is  $N$  and the critical value is  $N_* = 2$ .

it is possible to construct the function  $\eta_i(d, N)$  describing how the  $i$ -th scaling solution depends on the dimension  $d$  and on the number of components  $N$ .

By this technique we can follow the evolution of the RG fixed-points as we vary  $d$  and  $N$ . For  $d > 4$  and for any  $N$  we find only the Gaussian fixed-point; at  $d = 4$ , the upper critical dimension for the  $O(N)$ -models, the Wilson-Fisher fixed-point starts to branch away from the Gaussian fixed-point. In  $d = 3$  these fixed-points describe the known universality classes of the Ising, XY, Heisenberg and other models. Approaching  $d = 2$  one clearly observes that only the  $N = 1$  anomalous dimension continues to grow: for all other values of  $N \geq 2$  the anomalous dimension bends downward to become zero exactly when  $d = 2$ . This is a non-trivial fact, not evident from the structure of equation (1), telling us that only the  $O(N)$ -model with discrete symmetry ( $N = 1$ ) can have a second-order phase transition in two dimensions, while all the  $O(N)$ -models with continuous symmetry ( $N \geq 2$ ) cannot. This result, that here emerges from the RG analysis alone, is commonly known as the Mermin-Wagner-Hohenberg (MWH) theorem [4]. In Figure 1 we plotted the anomalous dimension as a function of  $d$  for the particular values  $N = 1, 2, 3, 4, 5, 10, 100$ ; this figure clearly represents the way in which the MWH manifests in the RG framework. This analysis can be seen as a RG confirmation of this important result and can be the starting point for a new rigorous proof of it. Note also that, as expected from the exact solution [9], the anomalous dimensions tend to zero for  $N \rightarrow \infty$ .

That there is no continuous phase transition in  $d = 2$  for the  $N \geq 2$  models can be confirmed by the analysis of the temperature critical exponent  $\nu_2(d, N)$ . This allows us to distinguish the Spherical model, related to the  $N \rightarrow \infty$  limit, from the Gaussian model, both having  $\eta = 0$ . We find [8] that only the  $N = 1$  model has a finite  $\nu_2$  in two dimensions, in all other models it diverges upon approaching  $d = 2$ , as is found in the  $N \rightarrow \infty$  case where

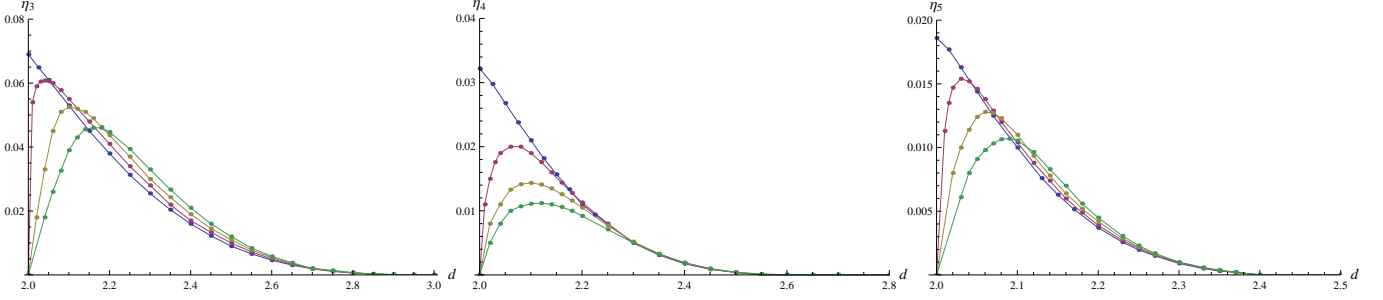


Figure 3: Anomalous dimensions  $\eta_i$  as a function of  $d$  for (from left) the tri-critical ( $i = 3$ ), tetra-critical ( $i = 4$ ) and penta-critical ( $i = 5$ ) scaling solutions for (from top at  $d = 2$ )  $N = 1, 2, 3, 4$ .

one knows exactly that  $\nu_2(d, \infty) = -\frac{1}{d-2}$ . An RG study by EAA techniques of this last case can be found in [9].

The critical case  $N = 2$  is known to have a distinguished behavior [10]. In this case one can observe all the distinctive properties of the Kosterlitz-Thouless phase transition by studying the properties of the RG flow; we refer to [11] for more details.

It is worth noting that our result for the anomalous dimension  $\eta_2(2, 1) = 0.234$  is in good agreement with the exact result  $\eta = 0.25$ , but we must mention that to provide an error estimate on this value one needs to go to higher orders of the derivative expansion [12].

To better discriminate between theories which can undergo a continuous phase transition in  $d_* = 2$  and those which cannot, we extend the analysis of scaling solutions to non-integer  $N$ ; in particular we want to see what happens around the critical value  $N_* = 2$ . The MWH theorem tells us that at  $d = d_*$  the “order parameter”  $\mathcal{O}(d, N) = \eta_2(d, N)/\eta_2(d, 1)$  is zero for  $N > N_*$  and non-zero for  $N < N_*$ ; but it tells us nothing about the “order” of the transition. For example, is the transition “continuous” in  $N$  or not? Figure 2 shows that the RG analysis can say a lot more about this. First, we see that  $\mathcal{O}(d, N)$  evolves continuously with  $N$  across  $N_*$ ; second, we see that  $\mathcal{O}(d, N)$  can be written in a scaling form around the transition point  $(d_*, N_*) = (2, 2)$ ; in particular we can write the following scaling relation:

$$\mathcal{O}(d_*, N) \sim \begin{cases} \left(\frac{N_* - N}{N_*}\right)^\Theta & N \rightarrow N_*^- \\ 0 & N \rightarrow N_*^+ \end{cases}, \quad (3)$$

where we introduced a new scaling exponent  $\Theta$ . A fit from the data displayed in Figure 2 gives the estimate  $\Theta \approx 0.98$  which is quite close to one. Relation (3) tells us how theory space deforms as we vary the external parameter  $N$ . An interesting question might be if relation (3) is universal, in the sense that the value of  $\Theta$  is independent of the details of the implementation of the RG procedure but rather describes an inner property of the set of theory spaces parametrized by  $N$ . One can make a reasoning similar to the above by keeping  $N$  fixed at its critical value but varying  $d$  around  $d_*$ :

$$\mathcal{O}(d, N_*) \sim \begin{cases} 0 & d \rightarrow d_*^- \\ \left(\frac{d - d_*}{d_*}\right)^{\frac{1}{\Delta}} & d \rightarrow d_*^+ \end{cases}. \quad (4)$$

In (4) we included the information, taken from [13], that  $\eta$  remains zero for  $N \geq N_*$  and  $d \leq d_*$ . A fit from the data displayed in Figure 1 gives the approximate value  $\Delta \approx 1.86$ . Finally, we remark that our analysis works when the first term on the rhs of (1) is non-zero, i.e. when  $\eta > 2 - d$ . This fact prevents us from performing a complete analysis in the range  $1 \leq d < 2$ , where indeed studies of  $O(N)$ -models on fractals have shown that the MWH is still valid [13].

#### *Multi-critical $O(N)$ -models in fractal dimension.*

When new universality classes appear by branching from the Gaussian fixed-point it is easy to determine the relative critical dimensions, since the argument based on canonical dimensions is valid. In particular, the  $i$ -th multi-critical scaling solution appears at the critical dimension  $d_{c,i} = 2 + \frac{2}{i-1}$ ; for integer values of  $i = 1, 2, 3, 4, \dots$  we find  $d_{c,i} = \infty, 4, 3, \frac{8}{3}, \frac{5}{2}, \frac{12}{5}, \frac{7}{3}, \frac{16}{7}, \frac{9}{4}, \dots$ . Each of these represents the upper critical dimension of the  $i$ -th multi-critical universality class. At these dimensions we see non-trivial fixed-points branching from the Gaussian even when  $N \geq 2$ . These fixed-points indeed correspond to multi-critical potentials of increasing order, i.e. with  $i$  minima when expressed in terms of the variable  $2\sqrt{\rho}$ .

The critical dimensions  $d_{c,i}$  accumulate at  $d = 2$  and thus one may naively expect to find, for any  $N$ , infinitely many universality classes in this dimension. Our analysis shows instead, see Figure 3 for the cases  $i = 3, 4, 5$  and  $N = 1, 2, 3, 4$ , that this happens only in the  $N = 1$  case, where the multi-critical fixed-points approach, in the limit  $d \rightarrow 2$ , the fixed-points representing the minimal-models found in CFT [3]. For any other  $N \geq 2$  we find that, consistently with the MWH theorem, the multi-critical scaling solutions, present in the range  $2 < d < 3$ , are instead absent in  $d = 2$ . This fact is a strong check of the general validity of the MWH theorem, which our analysis indicates is also applicable to multi-critical phase transitions. On the other side, we predict the existence of a whole family of  $O(N)$ -universality classes in fractal dimensions between two and three. To our knowledge these universality classes are new.

*The  $N \rightarrow 0$  limit.* In this last paragraph we study the  $N \rightarrow 0$  limit, that describes, through the correspondence

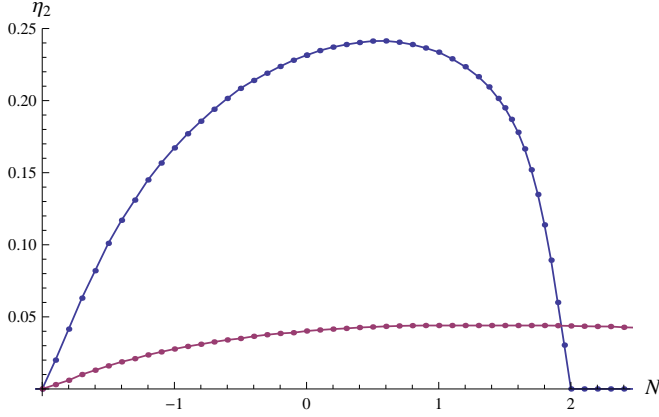


Figure 4:  $\eta_2$  as a function of  $N$  is continuous both in  $d = 2$  (upper curve) and in  $d = 3$  (lower curve). The limit  $N \rightarrow 0$  is well defined and both curves approach zero at  $N = -2$ , where the theory is exactly solvable with  $\eta = 0$ .

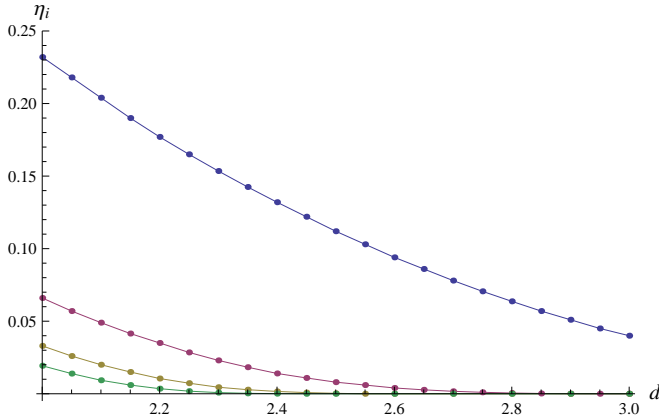


Figure 5: Anomalous dimensions  $\eta_i$  as a function of  $d$  for the first four  $N = 0$  multi-critical scaling solutions, i.e. for (from above)  $i = 2, 3, 4, 5$ .

found by De Gennes [14], the universality class of the self-avoiding random walk (SAW). Figure 4 shows the anomalous dimension as function of  $N$  in the interval between  $-2 \leq N \leq 2.5$  for the cases  $d = 2$  and  $d = 3$ . The figure shows that the anomalous dimensions are continuous in the whole range; this is an indication that the  $N \rightarrow 0$  limit is continuous, and thus well defined. Figure 4 also shows, interestingly, that both the  $d = 2$  and  $d = 3$  curves tend to zero as  $N \rightarrow -2$  where indeed the model is exactly solvable in both dimensions and has vanishing anomalous dimension [9].

As for the other integer values of  $N$ , we find multi-critical scaling solutions for  $N = 0$ . The interesting thing here is that these solutions survive in infinite number when  $d$  approaches two. A plot of the first four anomalous dimensions is shown in Figure 5; these are numerically very similar to those of the  $N = 1$  models (which can be found in Figure 1 and 3). This similarity is expected, as one may see by inspection of

Figure 4. Even if the anomalous dimension is not a relevant physical parameter in the SAW correspondence, we can use scaling relations to relate it to the physical critical exponents  $\nu$  and  $\gamma$  in order to make some checks. In  $d = 2$  one finds the exact values  $\nu = \frac{3}{4}$  and  $\gamma = \frac{43}{32}$  [15], and so  $\eta = 2 - \frac{\gamma}{\nu} = \frac{5}{24} \simeq 0.208$ ; we find  $\eta_2(2, 0) = 0.232$ . In  $d = 3$  one finds from Monte Carlo simulations the values  $\nu = 0.587$  and  $\gamma = 1.157$  [16], and so  $\eta = 2 - \frac{\gamma}{\nu} \simeq 0.029$ ; we find  $\eta_2(3, 0) = 0.04$ . As we said before, we cannot extend our method to  $d < 2$  to compare with exact SAW critical exponents found on fractals [17]. In any case, our analysis suggests that there is a countable family of  $O(N = 0)$ -universality classes in two dimensions. To our knowledge these are novel and may describe multi-critical phase transitions of some polymeric system. One may wonder if they are related to any CFT.

*Discussion and outlook.* In this Letter we studied how universality classes of linear scalar theories with  $O(N)$ -symmetry vary continuously with the dimension  $d$  and with the number of field components  $N$ . We followed the evolution of RG fixed-points, represented here by scaling solutions of the RG equation for the effective potential (1), through theory space as we varied these parameters. As in [3], even if all our analysis was based on the study of a simple ODE, we were able to observe a very rich behavior.

Above four dimensions and for any  $N$  we found, as expected, only the Gaussian universality class, while at  $d = 4$  we observed the Ising (or Wilson-Fisher) universality classes appear. In fractal dimension between two and three we found non-trivial fixed-points for all  $N$ : these are novel universality classes that in principle can be observed both in theoretical models on fractal lattices or in real physical systems.

Approaching two dimensions we observed the RG manifestation of the Mermin-Wagner-Hohenberg (MWH) theorem: only the  $N = 1$  universality classes survived down to  $d = 2$ , while all the  $N \geq 2$  ones disappeared. By considering  $(d, N)$  as real parameters near  $(2, 2)$  we found that the transition described by the MWH theorem, between theories that can undergo a continuous phase transition and theories that cannot, is continuous and that the anomalous dimension, which can be seen as analogous to the order parameter, can be written in scaling form at the critical point  $(2, 2)$ . Our analysis revealed how different theory spaces parametrized by  $N$  are related to each other; this information gives a deep RG understanding of the MWH theorem and can be used as the starting point for an extension of it.

Finally, we studied the  $N \rightarrow 0$  limit; we found that it is continuous around  $N = 0$  and we gave evidence of new  $O(N = 0)$ -universality classes in  $d = 2$ . These are analogous to the CFT minimal models series and may describe particular multi-critical transitions of polymeric systems.

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